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# Arithmetical properties of solutions of certain $q$ -difference equations

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In the present note we report certain results obtained by the author with Keijo Väänänen on the linear independence of the values of theta series and Tschakaloff functions.

## 1

Let  $\Theta(q, z)$  be the theta series defined by

$$(1) \quad \Theta(q, z) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \quad |q| < 1,$$

and let  $T_q(z)$  be the Tschakaloff function defined by

$$(2) \quad T_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{q^{\binom{n}{2}}}, \quad |q| > 1.$$

If  $qq = 1$ , then these functions are connected by the equation

$$(3) \quad \Theta(q, z) = T_{q^2}(q^{-1}z) + T_{q^2}(q^{-1}z^{-1}) - 1.$$

It was shown in [3] and [5] that Nesterenko's result [6] on the values of Ramanujan functions implies the transcendence of  $\Theta(q, 1)$  for all algebraic  $q$ ,  $0 < |q| < 1$  and that of  $T_q(1)$  for all algebraic  $q$ ,  $|q| > 1$ . However there are no such results for  $\Theta(q, z)$  with general algebraic  $q$ , even if we take some special value  $q = 1/q$ ,  $q \in \mathbf{Z} \setminus \{0, \pm 1\}$  for  $q$ . On the other hand, since the paper [8] of Tschakaloff there are a lot of works containing linear independence results on the values of  $T_q(z)$  and its derivatives, a good overview is given in [4]. In particular, it was proved in [8] if  $q$  is an integer in an imaginary quadratic field  $\mathcal{I}$ ,  $|q| > 1$ , and  $\alpha_1, \dots, \alpha_\ell$  are nonzero elements of  $\mathcal{I}$  satisfying  $\alpha_i/\alpha_j \notin q^{\mathbf{Z}}$  for all  $i \neq j$ , then the numbers  $1, T_q(\alpha_1), \dots, T_q(\alpha_\ell)$  are linearly independent over  $\mathcal{I}$ , see also [9] for some results on more general algebraic number

fields. By (3), one can obtain linear independence of  $1, \Theta(\mathbf{q}, \alpha_1), \dots, \Theta(\mathbf{q}, \alpha_\ell)$  under some extra conditions on  $\alpha_i$ , see [7], Korollar 2.

In the present note we are interested in the linear independence of the numbers

$$1, \Theta(\mathbf{q}_1, \alpha_1), \dots, \Theta(\mathbf{q}_\ell, \alpha_\ell)$$

with different  $\mathbf{q}_i$ . As far as we know there are no earlier results of this type.

## 2

In the following we assume that  $q \in \mathbb{Z} \setminus \{0, \pm 1\}$ .

**Theorem 1.** *Let  $\ell$  and  $L$  be positive integers with  $L \geq 2(\ell - 1)$ . Then there exists an effectively computable positive constant  $\gamma_1(\ell, L)$  such that, for any positive integers  $s_1, \dots, s_\ell$  satisfying*

$$\gamma_1(\ell, L) \leq s_1 < s_2 < \dots < s_\ell \leq s_1 + L/2$$

*and for any nonzero rational numbers  $\alpha_1, \dots, \alpha_\ell$  satisfying*

$$\alpha_i \notin -q^{s_i(1+2\mathbb{Z})} \quad (i = 1, \dots, \ell),$$

*the numbers*

$$1, \Theta(q^{-s_1}, \alpha_1), \dots, \Theta(q^{-s_\ell}, \alpha_\ell)$$

*are linearly independent over the rationals.*

This result is a consequence of the following result on the values of Tschakaloff functions.

**Theorem 2.** *Let  $\ell$  and  $L$  be positive integers with  $L \geq \ell - 1$ , and let  $m_1, \dots, m_\ell$  be positive integers. Then there exists an effectively computable positive constant  $\gamma_2(m, L)$  with  $m = m_1 + \dots + m_\ell$  such that, for any positive integers  $s_1, \dots, s_\ell$  satisfying*

$$\gamma_2(m, L) \leq s_1 < s_2 < \dots < s_\ell \leq s_1 + L$$

*and for any nonzero rational numbers  $\alpha_{ij}$  ( $i = 1, \dots, \ell$ ;  $j = 1, \dots, m_i$ ) satisfying*

$$\alpha_{ij_1}/\alpha_{ij_2} \notin q^{s_i\mathbb{Z}} \quad (i = 1, \dots, \ell; j_1 \neq j_2),$$

the numbers

$$1, T_{q^{s_i}}(\alpha_{ij}) \quad (i = 1, \dots, \ell; j = 1, \dots, m_i)$$

are linearly independent over the rationals.

We here deduce Theorem 1 from Theorem 2. Denoting  $q_i = q^{s_i}$ , we have by (3)

$$\Theta(q_i^{-1}, \alpha_i) = T_{q_i^2}(q_i^{-1}\alpha_i) + T_{q_i^2}(q_i^{-1}\alpha_i^{-1}) - 1 \quad (i = 1, \dots, \ell).$$

Since

$$\frac{q_i^{-1}\alpha_i}{q_i^{-1}\alpha_i^{-1}} = \alpha_i^2 \notin (q_i^2)^{\mathbf{Z}} \iff \alpha_i \notin \pm q_i^{\mathbf{Z}},$$

the statement of Theorem 1 follows from Theorem 2 under the right-hand conditions in the above. Therefore, our task is to relax these conditions into the conditions given in the theorem. To this aim let us fix  $i$  and assume  $\alpha_i \in \pm q_i^{\mathbf{Z}}$ . Then we have  $\alpha_i = \epsilon q_i^{-n}$  for some nonnegative integer  $n$  assuming  $|\alpha_i| \leq 1$  without loss of generality, where  $\epsilon$  is 1 or  $-1$ . (We may replace the role of  $\alpha_i$  by that of  $\alpha_i^{-1}$  if necessary.) By denoting  $\beta_i = q_i^{-1}\alpha_i$  this gives  $q_i^{-1}\alpha_i^{-1} = \beta_i q_i^{2n}$ . Since the repeated application of the functional equation

$$T_{q_i^2}(q_i^2 z) = q_i^2 z T_{q_i^2}(z) + 1$$

for  $T_{q_i^2}(z)$  implies

$$T_{q_i^2}(q_i^{2n} z) = q_i^{2\binom{n+1}{2}} z^n T_{q_i^2}(z) + P_{in}(z), \quad P_{in}(z) \in \mathbf{Z}[z],$$

on noting  $\beta_i = \epsilon q_i^{-(n+1)}$ , we have

$$T_{q_i^2}(q_i^{-1}\alpha_i^{-1}) = q_i^{2\binom{n+1}{2}} \beta_i^n T_{q_i^2}(\beta_i) + P_{in}(\beta_i) = \epsilon^n T_{q_i^2}(\beta_i) + P_{in}(\beta_i).$$

Hence,

$$\Theta(q_i^{-1}, \alpha_i) = \begin{cases} P_{in}(\beta_i) - 1, & \epsilon = -1 \text{ and } n \text{ is odd,} \\ 2T_{q_i^2}(\beta_i) + P_{in}(\beta_i) - 1, & \text{otherwise,} \end{cases}$$

which completes the deduction of Theorem 1 from Theorem 2.

For the proof of Theorem 2 we use the method given in [2] and [10]. More precisely, we define

$$f_{ij}(z) = \sum_{n=0}^{\infty} \frac{z^{s_i n}}{q^{s_i \binom{n+1}{2}}} \alpha_{ij}^n \quad (i = 1, \dots, \ell; j = 1, \dots, m_i),$$

which satisfy the  $q$ -difference equations

$$\alpha_{ij} z^{s_i} f_{ij}(z) = f_{ij}(qz) - 1.$$

Since  $f_{ij}(q) = T_{q^{s_i}}(\alpha_{ij})$ , the statement of Theorem 2 follows from the linear independence over the rationals of the numbers

$$1, f_{ij}(q) \quad (i = 1, \dots, \ell; j = 1, \dots, m_i).$$

To prove this statement we construct Padé-type approximations of the second kind for  $f_{ij}(z)$  taking the conditions  $s_1 < s_2 < \dots < s_\ell$  into account (see [1] for details).

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